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Again, the normal at  $P$  to the path of  $P$ ,  $\rho = f(\theta)$ , must also pass through  $C$ . Hence the angle  $OPC$  is the complement of the angle  $\psi$  between  $OP$  and the tangent at  $P$ . Hence,

$$\tan OPC = \cot \psi = \frac{d\rho}{\rho d\theta}.$$

(See any book on Calculus.) Therefore, from the right triangle  $COP$ , we have

$$OC = \rho \tan OPC = \frac{d\rho}{d\theta} = \frac{d}{d\theta} f(\theta).$$

Also solved by J. B. REYNOLDS and W. E. CEDERBERG.

**257 (Number Theory).** Proposed by LOUIS O'SHAUGHNESSY, University of Pennsylvania.

Find a general expression for the number of positive integers from 1 to  $10^t$ , inclusive, every one of which contains the figure 9 exactly  $r$  times ( $0 \leq r \leq t$ ).

#### SOLUTION BY THE PROPOSER.

In the case of the integers from 1 to 10, we have nine which do not contain the figure 9 and one which contains one 9. This shall be indicated by the expression  $9 + 1$ .

In the case of  $10^2$ , the number of integers, which do not contain 9, is  $9 \cdot 9$ , or  $9^2$ ; which contain one 9, is  $9 \cdot 1 + 9$ , or  $2 \cdot 9$ ; which contain two 9's, is 1, and we have the expansion of

$$(9 + 1)^2 = 9^2 + 2 \cdot 9 + 1.$$

For  $10^3$ , we have  $9 \cdot 9^2$ ,  $9 \cdot 2 \cdot 9 + 9^2$ ,  $9 \cdot 1 + 2 \cdot 9$ , and 1, or  $9^3 + 3 \cdot 9^2 + 3 \cdot 9 + 1$ .

Then, for  $10^k$ , assume the expansion of  $(9 + 1)^k$ , or

$$9^k + \binom{k}{1} 9^{k-1} + \binom{k}{2} 9^{k-2} + \dots + \binom{k}{n-1} 9^{k-(n-1)} + \binom{k}{n} 9^{k-n} + \dots + \binom{k}{k-1} 9 + 1.$$

For  $10^{k+1}$  we reason as follows: The number of integers which do not contain 9 is  $9 \cdot 9^k$ , or  $9^{k+1}$ ; which contain one 9, is  $9 \cdot \binom{k}{1} 9^{k-1} + 9^k$ , or  $\binom{k+1}{1} 9^k$ ; which contain two 9's, is  $9 \cdot \binom{k}{2} 9^{k-2} + \binom{k}{1} 9^{k-1}$  or  $\binom{k+1}{2} 9^{k-1}$ , and which contain  $n$  9's, is

$$9 \cdot \binom{k}{n} 9^{k-n} + \binom{k}{n-1} 9^{k-(n-1)} = [\binom{k}{n} + \binom{k}{n-1}] 9^{k-n+1} = \binom{k+1}{n} 9^{k+1-n}.$$

Hence, we have, for  $10^{k+1}$ , the expansion of  $(9 + 1)^{k+1}$ , or

$$9^{k+1} + \binom{k+1}{1} 9^k + \dots + \binom{k+1}{n} 9^{k+1-n} + \dots + \binom{k+1}{k} 9 + 1.$$

Now, the derived expression holds for  $k = 2$  and for  $k = 3$ ; hence it holds for all positive integral values of  $k$ .

Therefore, the general expression required is  $\binom{t}{r} 9^{t-r}$ .

Also solved by HORACE OLSON, H. C. FEEMSTER, C. C. YEN, and N. P. PANDYA.

**258 (Number Theory).** Proposed by A. A. BENNETT, University of Texas.

Find a recursion formula in terms of binomial coefficients for  $a_n$ , where the  $a$ 's are defined by the condition that the persymmetric determinants

$$\begin{vmatrix} a_0 & a_1 & a_2 & \cdot & \cdot \\ a_1 & a_2 & \cdot & \cdot & \cdot \\ a_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{n-1} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1 & a_2 & a_3 & \cdot & \cdot \\ a_2 & a_3 & \cdot & \cdot & \cdot \\ a_3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_n \end{vmatrix}$$

are each equal to unity for every positive integer  $n$ .

SOLUTION BY C. F. GUMMER, Queen's University, Kingston, Ont.

Though this solution does not directly involve binomial coefficients, yet by finding the value of  $a_n$  it may be considered to dispose of the problem sufficiently.

The given conditions show that  $a_0 = a_1 = 1$ , and that the other  $a$ 's may be found in succession uniquely from equations in which they appear with the coefficient unity. The  $a$ 's being determinate there exists a sequence  $x_0, x_1, \dots$  such that

$$(1) \quad a_n = a_{n-1}x_0 + a_{n-2}x_1 + \dots + a_0x_{n-1}, \quad n = 1, 2, \dots;$$

for the first  $n$  equations of (1) have a determinant equal to unity.

If we apply the substitutions (1) to the last row of

$$\begin{vmatrix} a_0 & a_1 & a_2 & \cdot & \cdot \\ a_1 & a_2 & \cdot & \cdot & \cdot \\ a_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & a_{n+1} & \cdot & \cdot & a_{2n} \end{vmatrix},$$

and simplify by means of the other rows, the last row becomes

$$0, a_0x_n, a_0x_{n+1} + a_1x_n, a_0x_{n+2} + a_1x_{n+1} + a_2x_n, \dots$$

With similar treatment, the preceding row becomes

$$0, a_0x_{n-1}, a_0x_n + a_1x_{n-1}, \dots,$$

and so for all but the first row. On simplifying by columns we get, since  $a_0 = 1$ ,

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdot & \cdot \\ x_2 & x_3 & \cdot & \cdot & \cdot \\ x_3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & x_{2n-1} \end{vmatrix} = 1.$$

A like treatment of the other determinant gives

$$\begin{vmatrix} x_0 & x_1 & x_2 & \cdot & \cdot \\ x_1 & x_2 & \cdot & \cdot & \cdot \\ x_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & x_{2n} \end{vmatrix} = 1.$$

Hence,  $x_2, x_3, \dots$  are defined in terms of  $x_0, x_1$  in the same way as  $a_2, a_3, \dots$  in terms of  $a_0, a_1$ .

Also  $a_0 = a_1 = 1, a_2 = 2$ .

Hence,  $x_0 = x_1 = 1, x_2 = 2$ , by direct calculation.

Hence,  $x_n = a_n$ . Hence, (1) becomes

$$(2) \quad a_n = a_{n-1}a_0 + a_{n-2}a_1 + \dots + a_0a_{n-1}, \quad n = 1, 2, \dots$$

To calculate  $a_n$ , we infer from (2) that the coefficient of  $t^n$  in  $u \equiv a_0 + a_1t + a_2t^2 + \dots$  is equal to the coefficient of  $t^{n-1}$  in  $u^2$ , when  $n = 1, 2, \dots$ .

Hence,  $(u - 1)/t = u^2$ .

Hence,  $u = 1/2t - \sqrt{1 - 4t}/(2t)$ , the minus sign being necessary to make  $u$  a series in positive powers of  $t$ .

Hence, the coefficient  $a_n$  of  $t^n$  in  $u$  equals  $\frac{|2n|}{n(n+1)}.$